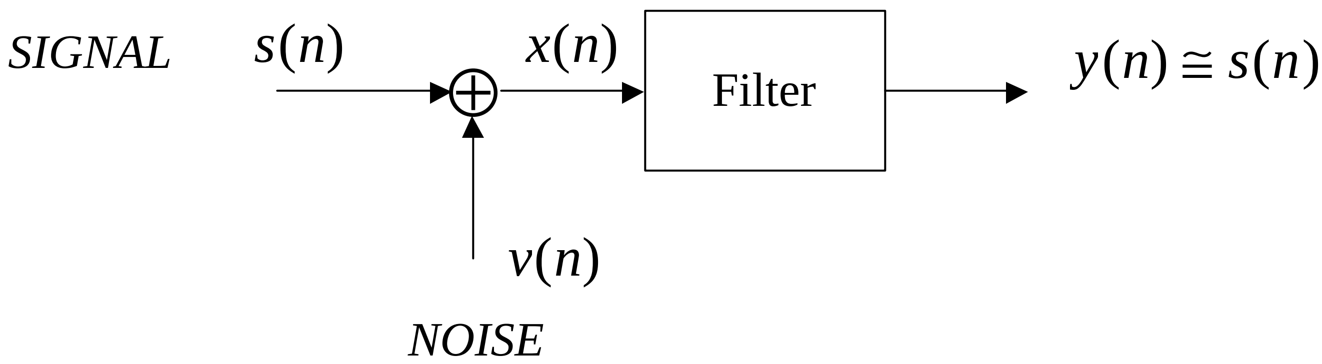
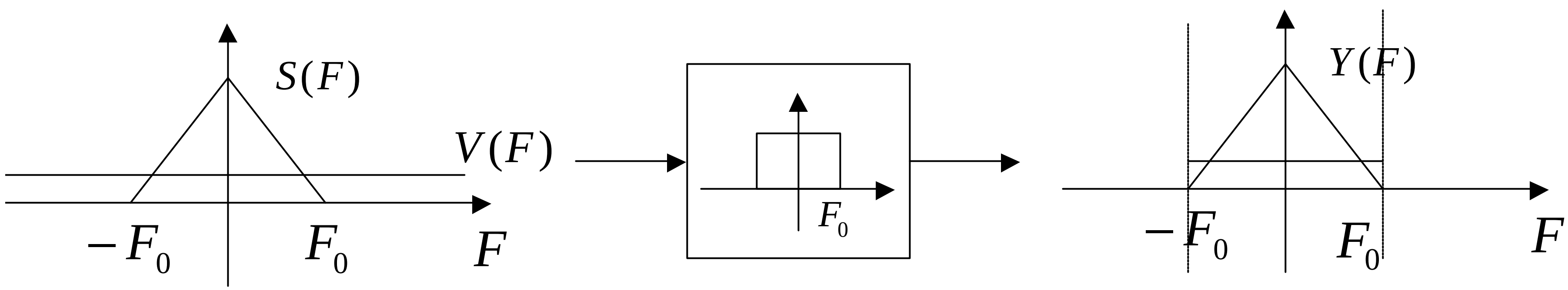
Ideal Filters

One of the reasons why we design a filter is to remove disturbances

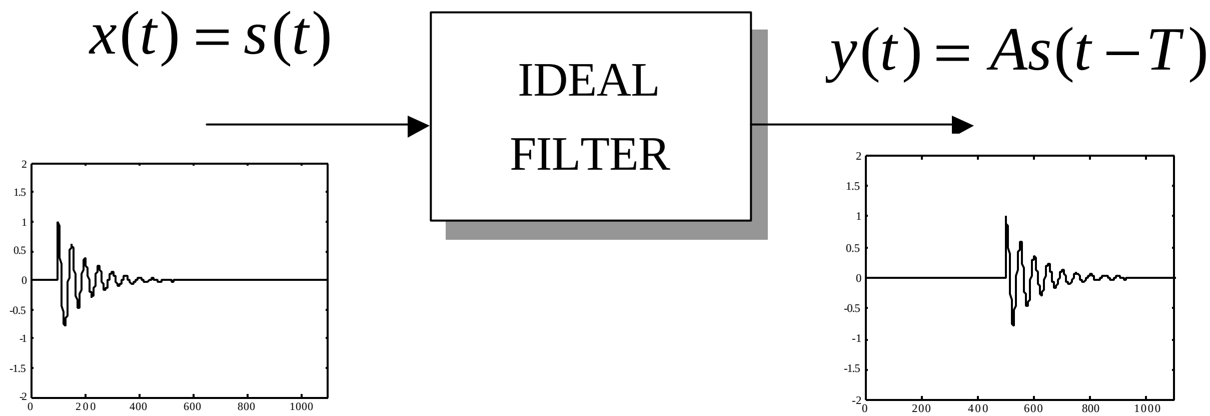


We discriminate between signal and noise in terms of the frequency spectrum



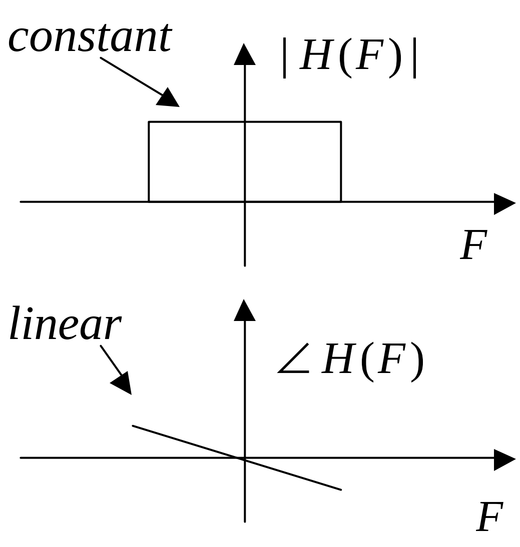
Conditions for Non-Distortion

Problem: *ideally we do not want the filter to distort the signal we want to recover.*



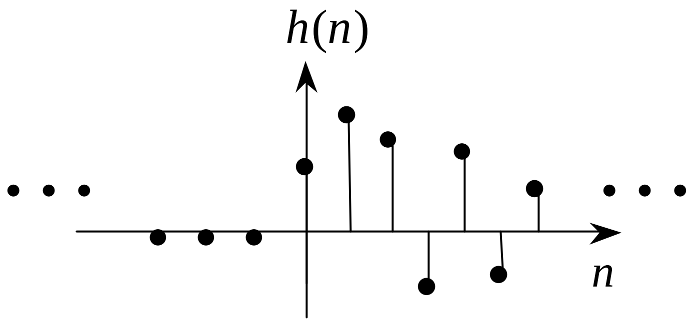
Same shape as s(t), just scaled and delayed.

Consequence on the Frequency Response:



$$H(F) = \begin{cases} Ae^{-j2pFT} & \text{if } F \text{ is in the passband} \\ 0 & \text{otherwise} \end{cases}$$

For *real time* implementation we also want the filter to be <u>causal</u>, ie.



$$h(n) = 0 \;\; \text{for } n < 0$$

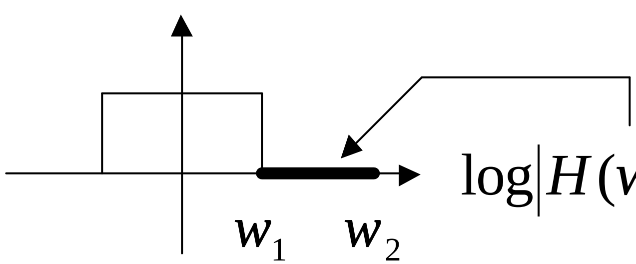
since

$$y(n) = \sum_{k=0}^{+\infty} h(k) \underbrace{x(n-k)}_{\text{past values only}}$$

<u>FACT</u> (Bad News!): *by the <u>Paley-Wiener Theorem</u>, if h(n) is causal and with finite energy,*

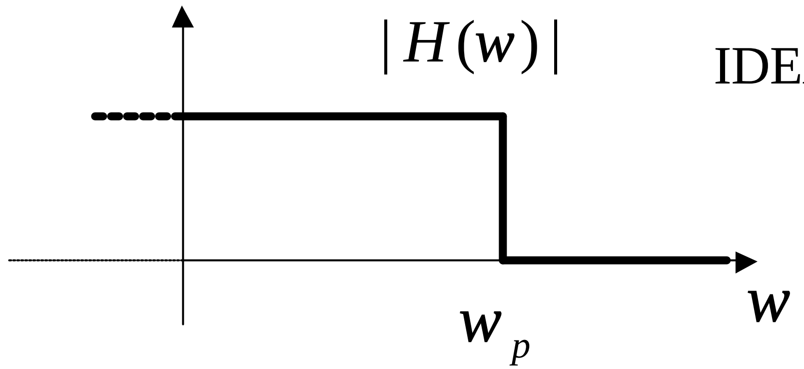
$$\int_{-p}^{+p} \left| \ln |H(w)| \right| dw < +\infty$$

ie $H(w)$ cannot be zero on an interval, therefore it $\boxed{\textbf{cannot be ideal.}}$



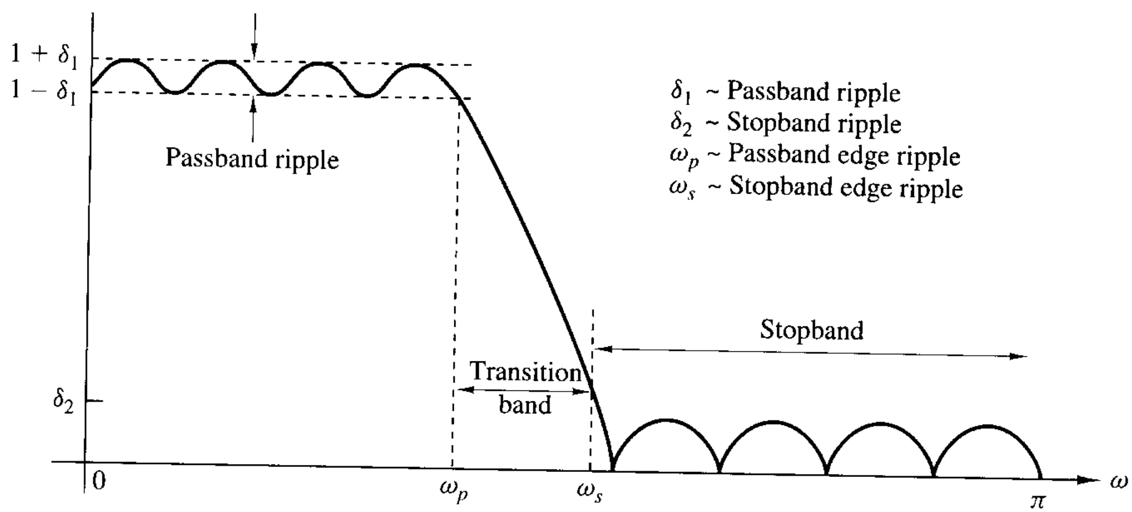
$$\log |H(w)| = \log(0) = -\infty \quad \Rightarrow \quad \int_{w_1}^{w_2} \left| \log |H(w)| \right| dw = +\infty$$

Characteristics of Non Ideal Digital Filters



IDEAL

Positive freq. only



NON IDEAL

Two Classes of Digital Filters:

a) Finite Impulse Response **(FIR),** non recursive, of the form

$$y(n) = h(0)x(n) + h(1)x(n-1) + ... + h(N)x(n-N)$$

With N being the order of the filter.

Advantages: always stable, the phase can be made exactly linear, we can approximate any filter we want;

Disadvantages: we need a lot of coefficients (*N large*) for good performance;

b) Infinite Impulse Response **(IIR)**, recursive, of the form

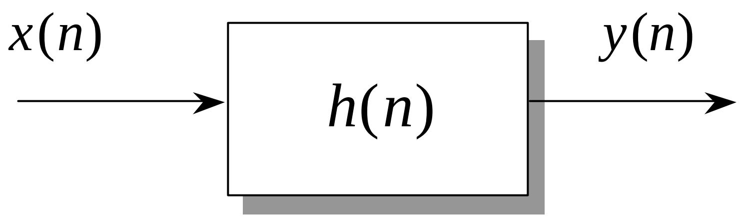
$$y(n) + a_1 y(n-1) + ... + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + ... + b_N x(n-N)$$

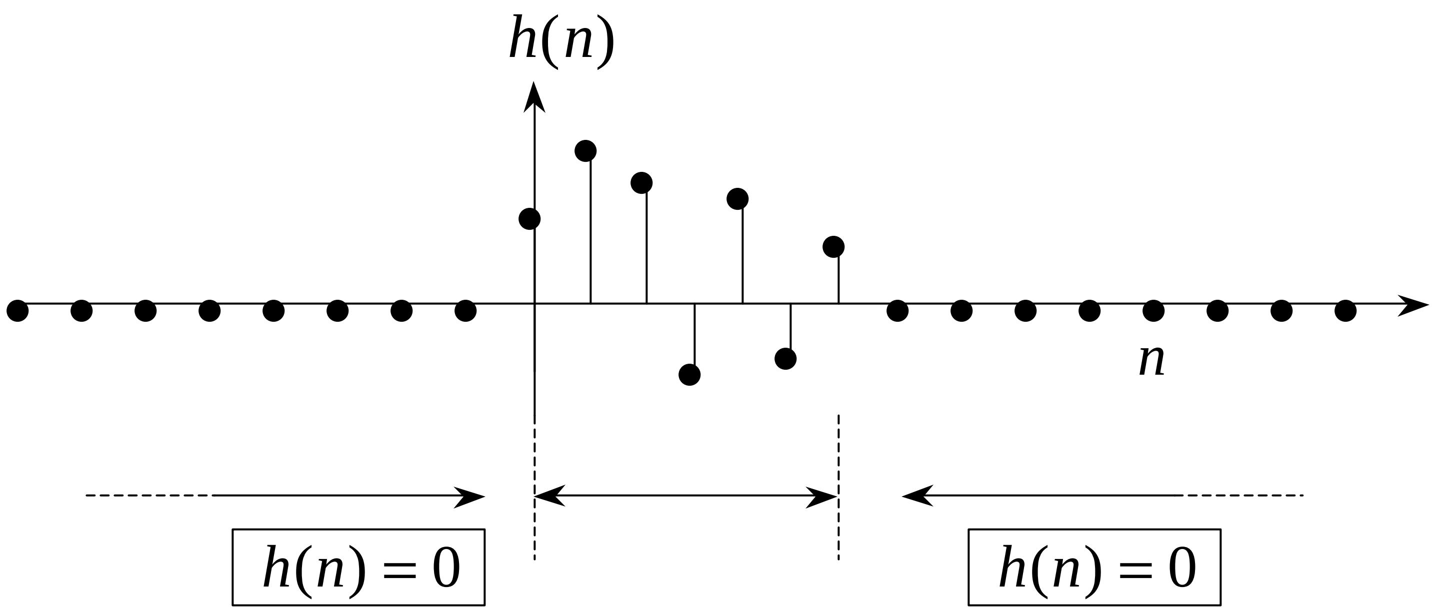
Advantages: very selective with a few coefficients;

Disadvantages: non necessarily stable, non linear phase.

Finite Impulse Response (FIR) Filters

Definition: a filter whose impulse response has finite duration.





<u>Problem</u>: *Given a desired Frequency Response* $H_d(w)$ *of the filter, determine the impulse response* $h(n)$.

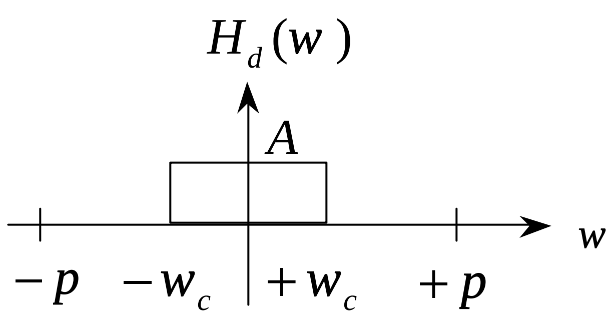
Recall: we relate the *Frequency Response* and the *Impulse Response* by the *DTFT:*

$$H_d(w) = DTFT\{h_d(n)\} = \sum_{n=-\infty}^{+\infty} h_d(n)e^{-jwn}$$

$$h_d(n) = IDTFT\{H_d(w)\} = \frac{1}{2p}\int_{-p}^{+p} H_d(w)e^{jwn}dw$$

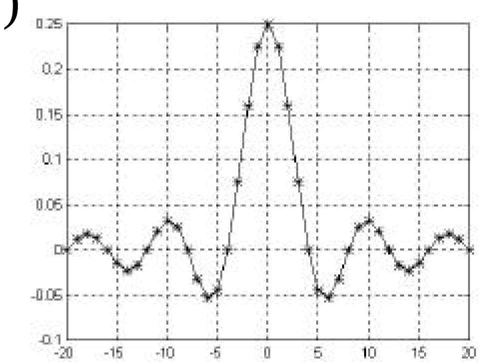
Example: <u>*Ideal Low Pass Filter*</u>

$$h_d(n) = \frac{1}{2p}\int_{-w_c}^{+w_c} Ae^{jwn}dw = \frac{\sin(w_c n)}{p\,n} = \frac{w_c}{p}\,\mathrm{sinc}\left(\frac{w_c}{p}n\right)$$



DTFT $\longleftrightarrow$

$h_d(n)$



n

$w_c = \frac{p}{4}$

Notice two facts:

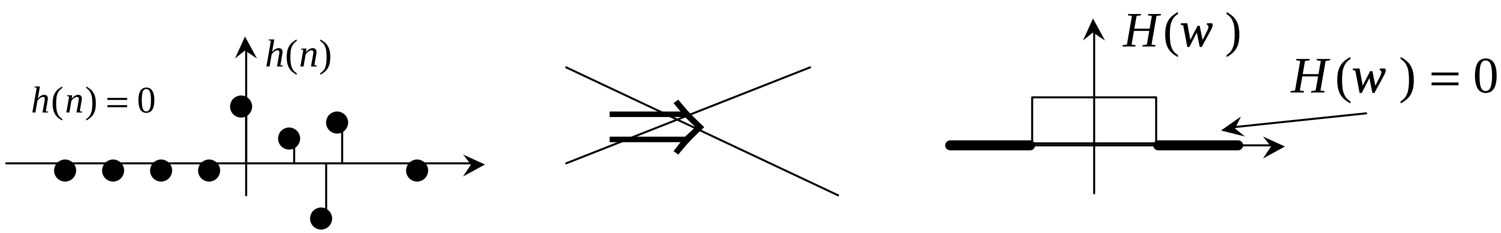
- the filter is <u>not causal</u>, i.e. the impulse response *h(n)* is non zero for *n<0*;

- the impulse response has <u>infinite</u> duration.

This is not just a coincidence. In general the following can be shown:

If a filter is <u>causal</u> then

- the frequency response cannot be zero on an interval;



- magnitude and phase are not independent, i.e. they cannot be specified arbitrarily

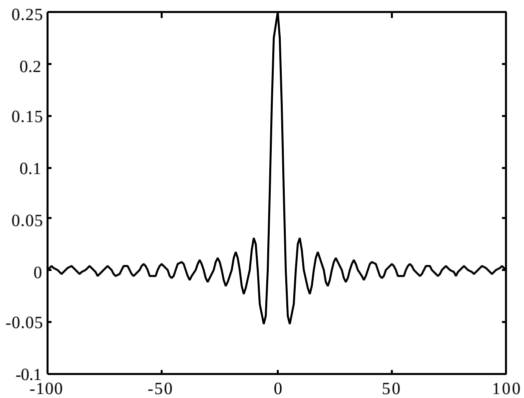
As a consequence: **an ideal filter cannot be causal.**

Problem: we want to determine a <u>*causal Finite Impulse Response (FIR)*</u> approximation of the ideal filter.

We do this by

a) <u>Windowing</u>

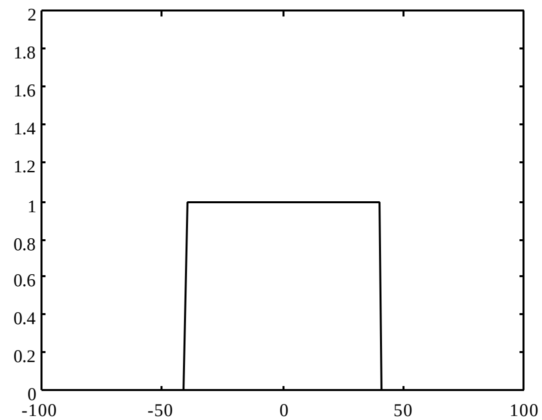
$h_d(n)$



infinite impulse response
(ideal)

$\times$

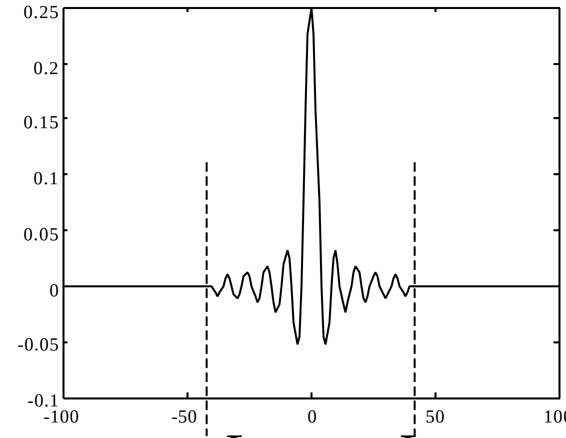
rectangular window



$-L \qquad L$

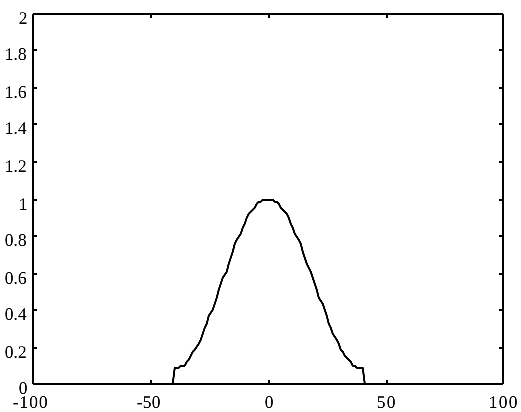
$=$

$h_w(n)$



$-L \qquad L$

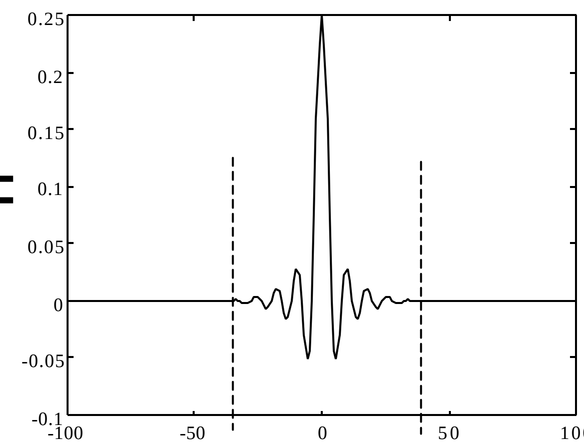
hamming window



$-L \qquad L$

$=$

$h_w(n)$

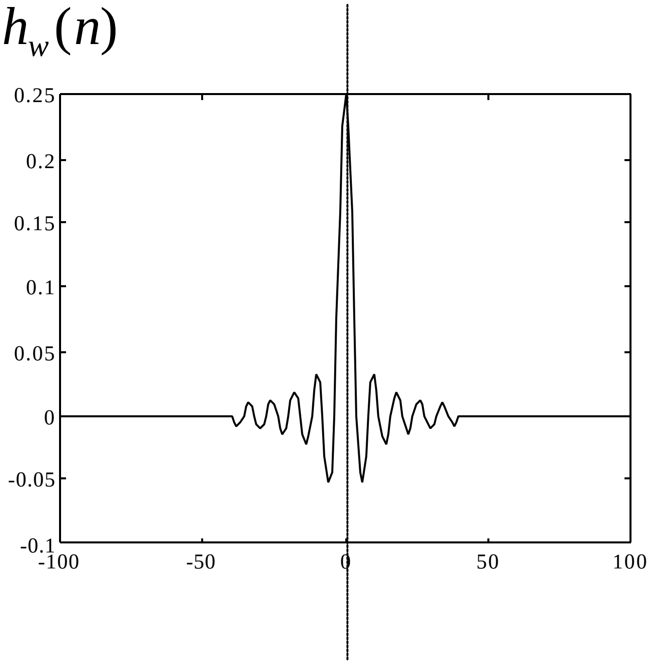


$-L \qquad L$

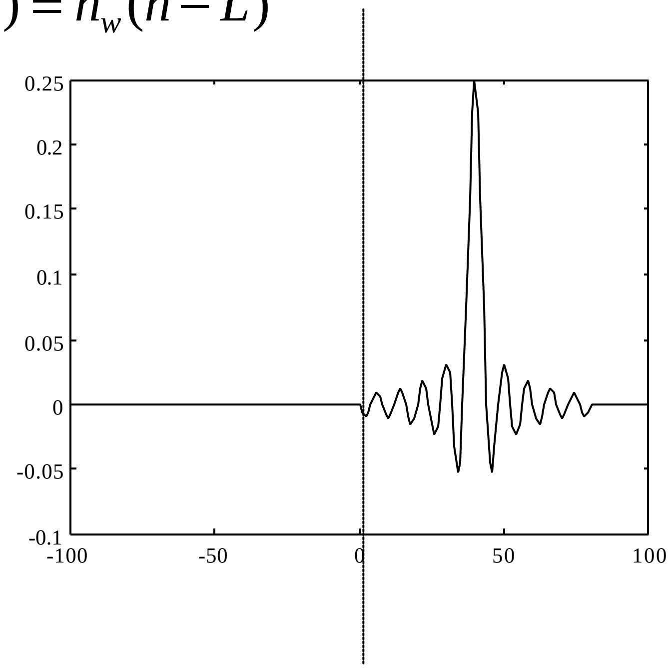
finite impulse response

b) <u>Shifting</u> in time, to make it causal:

$h_w(n)$

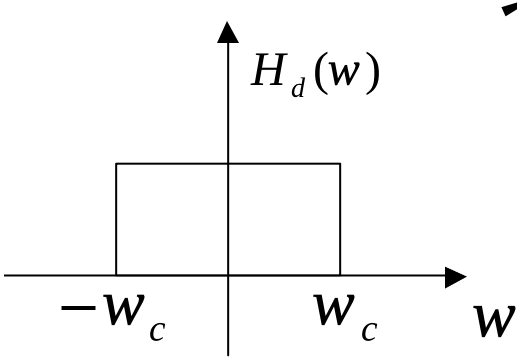


$h(n) = h_w(n - L)$

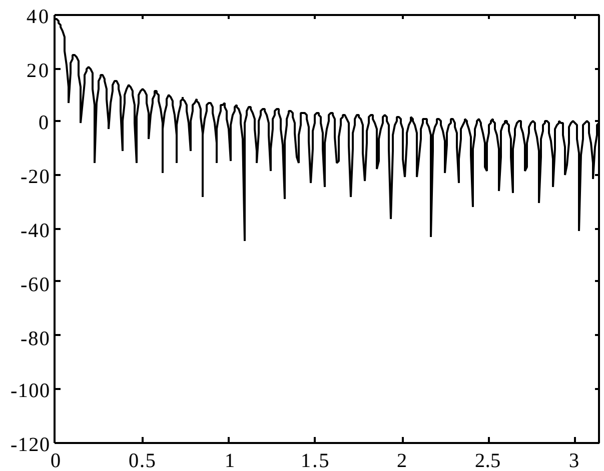


Effects of windowing and shifting on the frequency response of the filter:

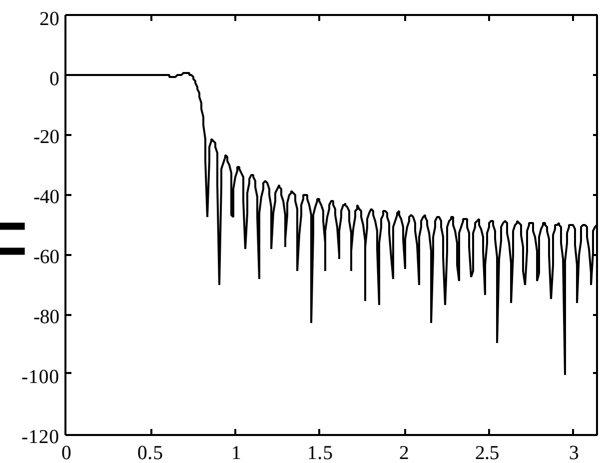
a) Windowing: since $h_w(n) = h_d(n)w(n)$ then $H_w(w) = \frac{1}{2p} H_d(w) * W(w)$



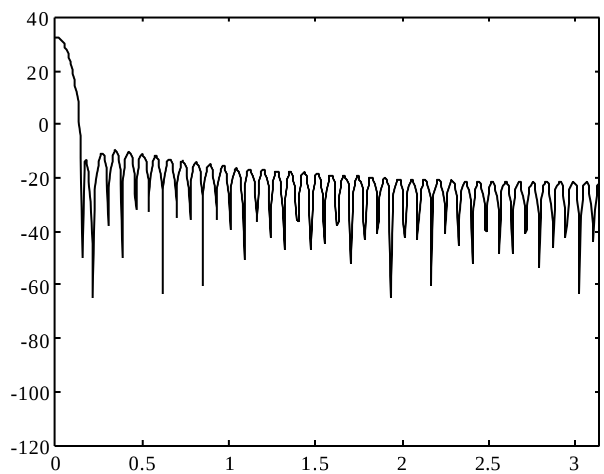
$|W(w)|$ **_rectangular window_** $|H_w(w)|$



$*$ $=$



hamming window



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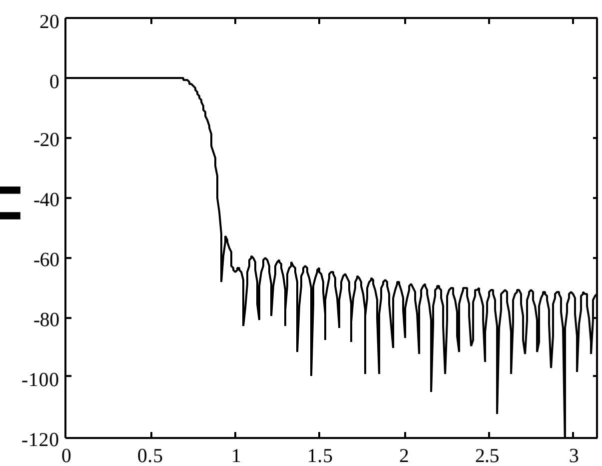
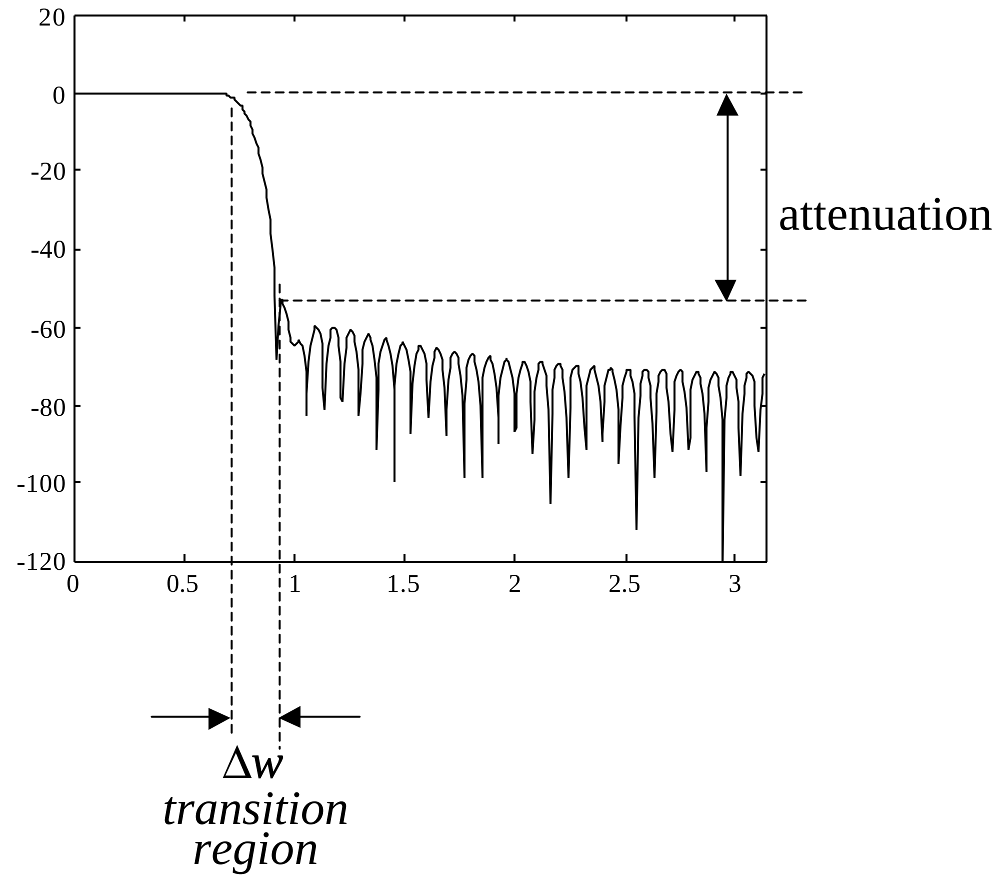
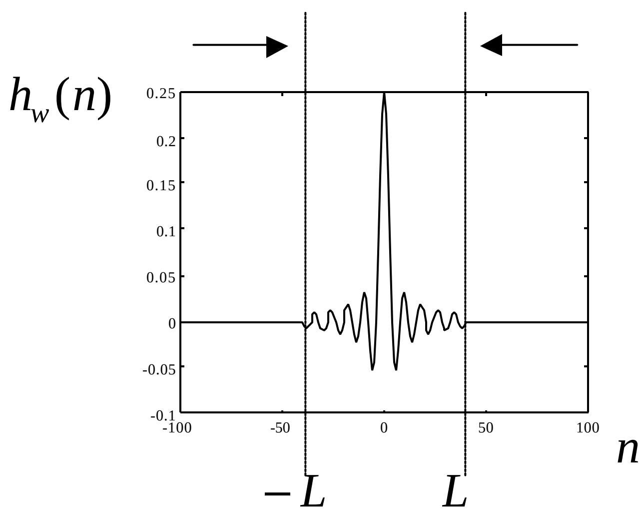


For different windows we have different values of the transition region and the attenuation in the stopband:

	Δw	attenuation
Rectangular	$4p\ /\ N$	-13dB
Bartlett	$8p\ /\ N$	-27dB
Hanning	$8p\ /\ N$	-32dB
Hamming	$8p\ /\ N$	-43dB
Blackman	$12p\ /\ N$	-58dB



with $N = 2L+1$



Effect of windowing and shifting on the frequency response:

b) shifting: since $h(n) = h_w(n-L)$ then $H(w) = H_w(w)e^{-jwL}$

Therefore

$$|H(w)| \;=\; |H_w(w)| \quad \text{no effect on the magnitude,}$$

$$\angle \mathrm{H}(w) = \angle \mathrm{H_w}(w) - wL \quad \text{shift in phase.}$$

See what is $\angle H_w(w)$.

For a Low Pass Filter we can verify the symmetry $h_w(n) = h_w(-n)$. Then

$$H_w(w) = \sum_{n=-\infty}^{+\infty} h_w(n) e^{-jwn} = h_w(0) + 2\sum_{n=1}^{+\infty} h_w(n)\cos(w\,n)$$

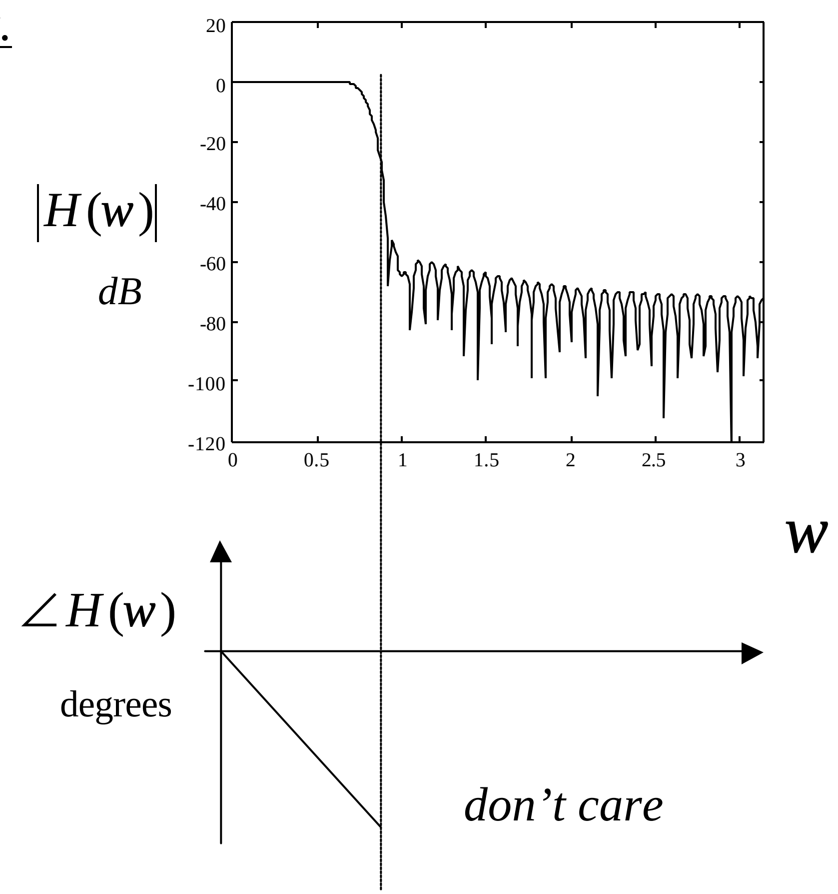
real for all w. Then

$$\angle H_w(w) = \begin{cases} 0 & \text{in the passband;} \\ don't \quad care, & \text{otherwise} \end{cases}$$

The phase of FIR low pass filter:

$$\angle H(w) = -wL \text{ in the passband;}$$

Which shows that it is a *<u>Linear Phase Filter.</u>*

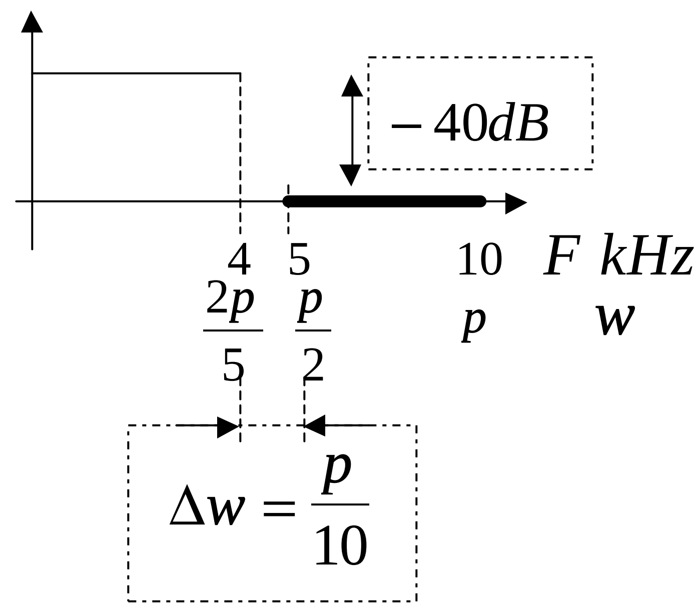


Example of Design of an FIR filter using Windows:

<u>*Specs:*</u> *Pass Band 0 - 4 kHz*

Stop Band > 5kHz with attenuation of at least 40dB

Sampling Frequency 20kHz



Step 1: translate specifications into digital frequency

Pass Band $0 \to 2\pi\, 4/20 = 2\pi/5\, rad$

Stop Band $2\pi\, 5/20 = \pi/2 \to \pi\ rad$

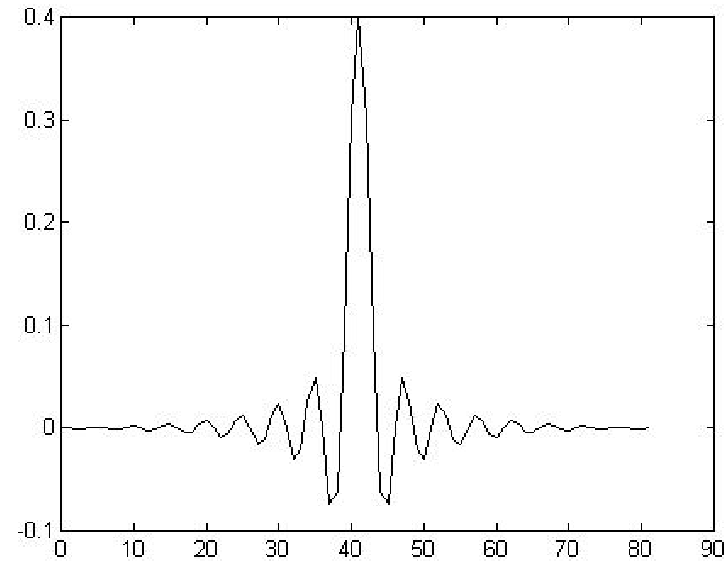
Step 2: from pass band, determine ideal filter impulse response

$$h_d(n) = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{\pi} n\right) = \frac{2}{5}\,\mathrm{sinc}\left(\frac{2n}{5}\right)$$

Step 3: from desired attenuation choose the window. In this case we can choose the *hamming window*;

Step 4: from the transition region choose the length *N* of the impulse response. Choose an odd number *N* such that:

$$\frac{8p}{N} \leq \frac{p}{10}$$

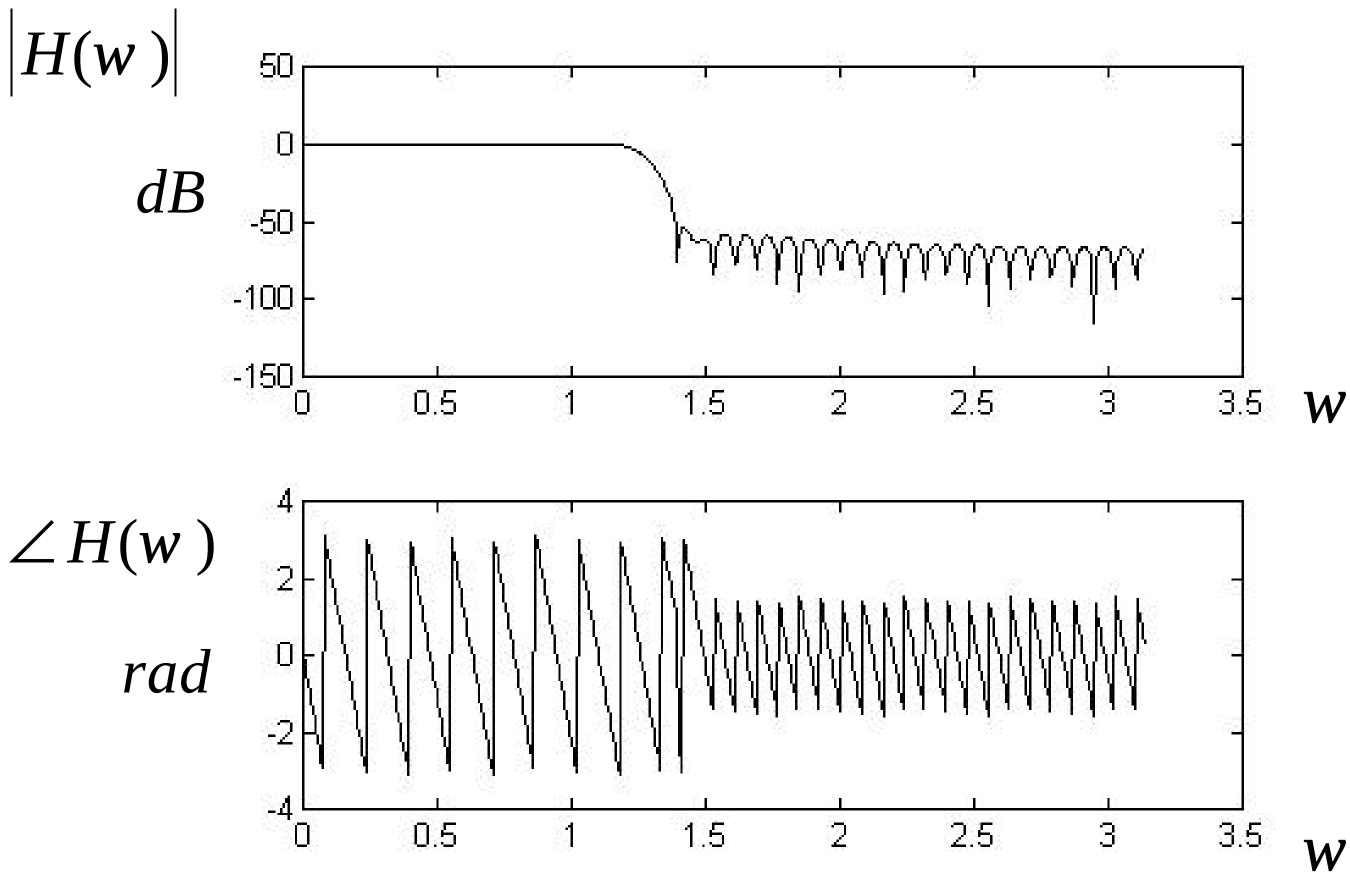


So choose *N=81* which yields the shift *L=40*.

Finally the impulse response of the filter

$$h(n) = \begin{cases} \frac{2}{5}\,\mathrm{sinc}\left(\frac{2(\mathrm{n}-40)}{5}\right)\left(0.54 - 0.46\cos\left(\frac{2p\,n}{80}\right)\right), & \text{if } \ 0 \leq n \leq 80, \\ 0 & \text{otherwise} \end{cases}$$

The Frequency Response of the Filter:



Example: design a digital filter which approximates a differentiator.

Specifications:

• *Desired Frequency Response:*

$$H_d(F) = \begin{cases} j2p\ F & \text{if} \quad -4kHz \le F \le +4kHz \\ 0 & \text{if} \qquad F > 5kHz \end{cases}$$

• *Sampling Frequency* $F_s = 20kHz$

• *Attenuation in the stopband at least 50dB.*

Solution.

Step 1. Convert to digital frequency

$$H_d(w) = H_d(F)\Big|_{F = wF_s/2p} = \begin{cases} jwF_s = j20{,}000w & \text{if } -\dfrac{2p}{5} \le w \le \dfrac{2p}{5} \\ 0 & \text{if } \dfrac{p}{2} < |w| \le p \end{cases}$$

Step 2: determine ideal impulse response

$$h_d(n) = IDTFT\left\{H_d(w)\right\} = \frac{1}{2p}\int_{-\frac{2p}{5}}^{\frac{2p}{5}} j20{,}000w\, e^{jwn}\,dw$$

From integration tables or integrating by parts we obtain $\int xe^{ax}dx = \dfrac{e^{ax}}{a}\left(x - \dfrac{1}{a}\right)$

Therefore

$$h_d(n) = \begin{cases} 20{,}000\left(\dfrac{4p}{5}\dfrac{\cos\left(\dfrac{2pn}{5}\right)}{n} - 2\dfrac{\sin\left(\dfrac{2pn}{5}\right)}{n^2}\right) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

Step 3. From the given attenuation we use the Blackman window. This window has a transition region region of $12p / N$. From the given transition region we solve for the complexity *N* as follows

$$\Delta w = \frac{p}{2} - \frac{2p}{5} = 0.1p \geq \frac{12p}{N}$$

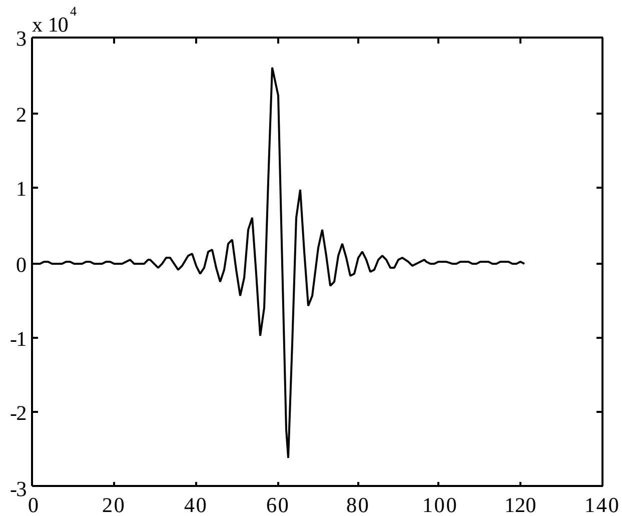
which yields $N \geq 120$ Choose it odd as, for example, *N=121*, *ie. L=60.*

Step 4. Finally the result

$$h(n) = 20{,}000\left(\frac{4p}{5}\frac{\cos\left(\frac{2p\,(n-60)}{5}\right)}{n-60} - 2\frac{\sin\left(\frac{2p\,(n-60)}{5}\right)}{(n-60)^2} \right)\left(0.42 - 0.5\cos\left(\frac{2pn}{120}\right) + 0.08\cos\left(\frac{4pn}{120}\right)\right)$$

$$\text{for } 0 \leq n \leq 120$$

Impulse response *h(n)*



Frequency Response